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Scale-covariant field theories: IV. Stability

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Abstract. At a 'semiclassical' level changing from translation-invariant to scale-covariant measures introduces the likelihood of instability. We present two mechanisms whereby, because of the quantum effects, instability is avoided.

1. Introduction

Scale-covariant field theories have been proposed by Klauder (1979a, b, 1981, and references therein) as a solution to the problem of non-renormalisability.

For simplicity let us restrict ourselves to the theory of a single scalar field. The generating functionals that have to be evaluated are (for a Euclidean theory) formally written as

$$Z'[h]=\int \mathscr{D}'[\varphi]\exp -\frac{1}{\hbar}\Big(A[\varphi]-\int h\varphi\Big) \tag{1.1}$$

where $A[\varphi]$ is the classical action

$$A[\varphi]=\int \mathrm{d}x[\tfrac{1}{2}(\nabla\varphi)^2+V(\varphi)] \tag{1.2}$$

and $\mathscr{D}'[\varphi]$ is the scale-covariant measure satisfying (Klauder 1981)

$$\mathscr{D}'[\Lambda\varphi]=F[\Lambda]\mathscr{D}'[\varphi] \qquad \text{for all } \Lambda,\ \Lambda(x)>0,\ \forall x. \tag{1.3}$$

In the previous two papers of this series (Ebbutt and Rivers 1982a, b, to be referred to as II and III, respectively) we have either (i) worked with the scale-covariant branching equations for the connected Green functions implied by Z' (as in II), or (ii) worked with the augmented formalism, in which (for $F=1$) the scale-*invariant* measure $\mathscr{D}'[\varphi]$ is re-expressed in terms of *translation-invariant* measures $\mathscr{D}[\varphi]$, $\mathscr{D}[\chi]$ as

$$\mathscr{D}'[\varphi]=\int \mathscr{D}[\varphi]\mathscr{D}[\chi]\exp -\frac{\eta}{2\hbar}\int \mathrm{d}x\,\varphi^2\chi^2 \tag{1.4}$$

(with suitable normalisation). That is, in terms of the more convenient translation-invariant measures, the action $A[\varphi]$ is replaced by

$$A'[\varphi,\chi]=A[\varphi]+\tfrac{1}{2}\eta\int \mathrm{d}x\,\varphi^2\chi^2. \tag{1.5}$$

The consequences of this were discussed in III.

In both approaches we have circumvented the fact that the scale-covariant measure $\mathscr{D}'[\varphi]$ is very singular. This is seen by writing $\mathscr{D}'[\varphi]$ formally as

$$\mathscr{D}'[\varphi]=\frac{\mathscr{D}[\varphi]}{\Pi_x|\varphi(x)|^\beta} \qquad 0<\beta\leqslant 1. \tag{1.6}$$

That is, for a given field configuration $\varphi(x)$, $\mathscr{D}'$ diverges at its *zeros*. If, instead of (1.4), we were to write† (formally)

$$\mathscr{D}'[\varphi]=\mathscr{D}[\varphi]\exp -\frac{\beta}{2}\delta(0)\int \mathrm{d}x \ln \varphi^2 \tag{1.7}$$

we see that, in terms of translationally invariant measures, we have replaced the potential $V(\varphi)$ in (1.2) by

$$V'(\varphi)=V(\varphi)+\tfrac{1}{2}\beta\hbar\delta(0)\ln\varphi^2. \tag{1.8}$$

As $\varphi\to 0$, the additional 'potential' $\frac{1}{2}\beta\hbar\delta(0)\ln\varphi^2$ becomes unbounded below (assuming some regularisation procedure). This suggests that the change of measure may make the theory intrinsically unstable. If this were the case it would invalidate the whole idea of scale covariance, and the possibility deserves to be considered seriously.

In this paper we shall argue that this unboundedness of the modified classical potential is not necessarily reflected in the more appropriate *effective potential* of the theory. To see this it is sufficient to examine the *pseudo-free* scalar theory in which $A[\varphi]$ describes a free field as

$$A[\varphi]=\int \mathrm{d}x[\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2]. \tag{1.9}$$

To demonstrate stability it is necessary to avoid $\hbar$ expansions. Two approaches suggest themselves. As a first indication of how instability can be avoided we shall examine the large-N limit of the $O(N)$-invariant pseudo-free theory, obtained by generalising (1.9) to N fields φ_i $(i=1,2,\ldots,N)$ (in the vector representation). As a second indication of how quantum effects can restore stability we adopt what is essentially a 'strong-coupling' expansion for a single pseudo-free scalar field, in which we expand in kinetic terms about the pseudo-free independent-value model (IVM) (Klauder 1975) with action

$$A_0[\varphi]=\tfrac{1}{2}\int \mathrm{d}x\, m_0^2\varphi^2. \tag{1.10}$$

This discussion of the Euclidean pseudo-free theory is the content of the next two sections. The lower boundedness of the Euclidean effective action that each implies is encouraging. However, the interpretation of the effective potential as an energy density is only strictly correct for the Minkowski theory, and this is the content of the third section of the paper.

The concluding section summarises our results.

† This can be made well defined (Kotecky and Preiss 1978) for the independent-value model in which the kinetic term is dropped and for which $\beta=1$.

2. The large-N limit of the pseudo-free effective potential

The expression (1.8) for the modified classical potential is not, in itself, proof of instability of the scalar theory. Rather, we need to calculate the effective potential $\mathcal{V}(\varphi)$, the constant field density of the effective action $\Gamma[\varphi]$ (the generating functional of one φ irreducible Green functions).

In general, we do not know how to compute $\mathcal{V}$. An expansion in $\hbar$ is clearly inappropriate. However, we have seen in III that a diagrammatic expansion of the 'hard-core' effect of the change of action, when expressed as in (1.5), leads naturally to a retention of the most singular diagrams. This occurs automatically in $1/N$ or mean-field expansions, and it is such an expansion that we shall consider here.

Consider the O(N)-invariant scale-covariant pseudo-free Euclidean theory of the scalar fields φ_i ($i = 1, 2, \ldots, N$) with generating functional

$$Z'[\boldsymbol{h}] = \int \mathscr{D}'[\boldsymbol{\varphi}] \exp -\frac{1}{\hbar}\int \mathrm{d}x[\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \tfrac{1}{2}m_0^2\boldsymbol{\varphi}^2 - \boldsymbol{h}\cdot\boldsymbol{\varphi}]. \tag{2.1}$$

We require that $\mathscr{D}'[\boldsymbol{\varphi}]$ be covariant under both O(N) rotations and under local O(N)-invariant scale transformations i.e.

$$\mathscr{D}'[\Lambda\boldsymbol{\varphi}] = F[\Lambda]\mathscr{D}'[\boldsymbol{\varphi}] \qquad \text{for all } \Lambda,\ \Lambda(x) > 0, \forall x. \tag{2.2}$$

In terms of the O(N)-invariant translation-invariant measure $\mathscr{D}[\boldsymbol{\varphi}] = \Pi_1^N \mathscr{D}[\varphi_i]$ we can formally write (up to normalisation)

$$\mathscr{D}'[\boldsymbol{\varphi}] = \prod_{i=1}^{N}\left[\frac{\mathscr{D}[\varphi_i]}{\Pi_x|N^{-1/2}\boldsymbol{\varphi}(x)|^{\beta}}\right] = \frac{\mathscr{D}[\boldsymbol{\varphi}]}{\Pi_x|N^{-1/2}\boldsymbol{\varphi}(x)|^{N\beta}} \tag{2.3}$$

whence

$$Z'[\boldsymbol{h}] = \int \prod_1^N \mathscr{D}[\varphi_i] \exp -\frac{1}{\hbar}\int \mathrm{d}x[\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \tfrac{1}{2}m_0^2\boldsymbol{\varphi}^2 + \tfrac{1}{2}\beta\hbar N\delta(0)\ln N^{-1}\boldsymbol{\varphi}^2 - \boldsymbol{h}\cdot\boldsymbol{\varphi}] \tag{2.4}$$

displaying the suspect $\ln \boldsymbol{\varphi}^2$ term.

We expect the path integral to be dominated by those configurations for which $\boldsymbol{\varphi}^2 = \mathrm{O}(N)$. To make this N dependence explicit we rewrite Z' as

$$\begin{aligned} Z'[\boldsymbol{h}] &= \int \prod_1^N \mathscr{D}[\varphi_i]\mathscr{D}[\sigma][\delta(\boldsymbol{\varphi}^2 - N\sigma)] \\ &\qquad \times \exp -\frac{1}{\hbar}\int \mathrm{d}x[\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \tfrac{1}{2}m_0^2\boldsymbol{\varphi}^2 + \tfrac{1}{2}\beta N\hbar\delta(0)\ln N^{-1}\boldsymbol{\varphi}^2 - \boldsymbol{h}\cdot\boldsymbol{\varphi}] \\ &= \int \prod_1^N \mathscr{D}[\varphi_i]\mathscr{D}[\sigma]\mathscr{D}[\alpha] \\ &\qquad \times \exp -\frac{1}{\hbar}\int \mathrm{d}x[\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \tfrac{1}{2}m_0^2\boldsymbol{\varphi}^2 + \tfrac{1}{2}\mathrm{i}\alpha(\boldsymbol{\varphi}^2 - N\sigma) + \tfrac{1}{2}\beta N\hbar\delta(0)\ln\sigma - \boldsymbol{h}\cdot\boldsymbol{\varphi}]. \end{aligned} \tag{2.5}$$

The Gaussian $\boldsymbol{\varphi}$ integrals can now be performed to give

$$Z'[\boldsymbol{h}] = \int \mathscr{D}[\sigma]\mathscr{D}[\alpha] \exp -\frac{1}{\hbar}\mathfrak{A}[\sigma, \alpha; \boldsymbol{h}] \tag{2.6}$$

where

$$\mathfrak{A}[\sigma, \alpha; \boldsymbol{h}] = -\tfrac{1}{2}\int \boldsymbol{h}\cdot(-\nabla^2 + m_0^2 + \mathrm{i}\alpha)^{-1}\boldsymbol{h} + \tfrac{1}{2}N\int (\beta\hbar\delta(0)\ln\sigma - \mathrm{i}\alpha\sigma)$$
$$+\tfrac{1}{2}N\hbar\,\mathrm{Tr}\ln(-\nabla^2 + m_0^2 + \mathrm{i}\alpha). \tag{2.7}$$

Each term in (2.7) is O(N). In the limit $N\to\infty$ ($\hbar$ fixed) we assume that the path integral $Z'[\boldsymbol{h}]$ is dominated by a single $\hbar$ dependent saddle point, given by the minimum of $\mathfrak{A}$.

In computing the effective potential $\mathscr{V}(\boldsymbol{\varphi})$ it is sufficient to take $\boldsymbol{h}$ to be a *constant vector.* If Ω denotes d-dimensional Euclidean space–time volume, we can write (for such $\boldsymbol{h}$)

$$Z'(\boldsymbol{h}) = \exp -\frac{\Omega}{\hbar}W(\boldsymbol{h}) = C\exp -\frac{\Omega}{\hbar}v(\boldsymbol{h}) \tag{2.8}$$

where $v(\boldsymbol{h})$ is obtained from (constant σ, α)

$$v(\sigma, \alpha; \boldsymbol{h}) = -\tfrac{1}{2}(m_0^2 + \mathrm{i}\alpha)^{-1}\boldsymbol{h}^2 + \tfrac{1}{2}N(\beta\hbar\delta(0)\ln\sigma - \mathrm{i}\alpha\sigma)$$
$$+\tfrac{1}{2}N\hbar\int\frac{\mathrm{d}^dk}{(2\pi)^d}\ln(k^2 + m_0^2 + \mathrm{i}\alpha) \tag{2.9}$$

on evaluating it at

$$\frac{\partial v}{\partial\sigma} = 0 = \frac{\partial v}{\partial\alpha}. \tag{2.10}$$

As before, v is O(N) and C, the determinant of fluctuations about the extremum (2.10), has $\ln C$ of O(1).

The Euclidean effective potential $\mathscr{V}(\boldsymbol{\varphi})$ is the Legendre transform of $W(\boldsymbol{h})$ and hence, in the large-N limit, of $v(\boldsymbol{h})$.

Two steps simplify the evaluation of $\mathscr{V}(\boldsymbol{\varphi})$. Firstly, we choose

$$m^2 = m_0^2 + \mathrm{i}\alpha$$

as a more convenient variable than α. Secondly, as an intermediate step, we introduce $\mathscr{V}(m^2, \sigma; \boldsymbol{\varphi})$, the Legendre transform of $v(m^2, \sigma; \boldsymbol{\varphi})$. That is, defining $\boldsymbol{\varphi}$ by

$$\varphi_i = -\frac{\partial v}{\partial h_i}(m^2, \sigma; \boldsymbol{h}) = m^{-2}h_i \tag{2.11}$$

we construct

$$\mathscr{V}(m^2, \sigma; \boldsymbol{\varphi}) = v(m^2, \sigma; \boldsymbol{h}(\boldsymbol{\varphi})) + \boldsymbol{\varphi}\cdot\boldsymbol{h}(\boldsymbol{\varphi}) \tag{2.12}$$

$$= \tfrac{1}{2}m^2\boldsymbol{\varphi}^2 + \tfrac{1}{2}\beta N\hbar\delta(0)\ln\sigma - \tfrac{1}{2}N(m^2 - m_0^2)\sigma + \tfrac{1}{2}N\hbar\int đk\,\ln(k^2 + m^2) \tag{2.13}$$

where $đk = (2\pi)^{-d}\,\mathrm{d}^dk$ (in d space–time dimensions).

The effective potential $\mathscr{V}(\boldsymbol{\varphi})$ is now obtained by imposing the constraint equations

$$0 = -\frac{2}{N}\frac{\partial\mathscr{V}}{\partial\sigma}(m^2, \sigma; \boldsymbol{\varphi})\bigg|_{\substack{m^2 = m^2(\boldsymbol{\varphi}^2)\\ \sigma = \sigma(\boldsymbol{\varphi}^2)}} = m^2(\boldsymbol{\varphi}^2) - m_0^2 - \beta\hbar\delta(0)\sigma(\boldsymbol{\varphi}^2)^{-1} \tag{2.14}$$

$$0=-\frac{2}{N}\frac{\partial \mathcal{V}}{\partial m^2}(m^2,\sigma;\varphi)\bigg|_{\substack{m^2=m^2(\varphi^2)\\ \sigma=\sigma(\varphi^2)}}=\sigma(\varphi^2)-\frac{\varphi^2}{N}-\hbar\int\frac{đk}{k^2+m^2(\varphi^2)}. \tag{2.15}$$

Equations (2.14) and (2.15) are identical to (2.10), on making the substitution (2.11).

We see that if we were to set $\hbar=0$ in (2.14) and (2.15) we would have $m^2=m_0^2$, $\sigma=N^{-1}\varphi^2$, whence

$$\mathcal{V}(\varphi)=\tfrac{1}{2}m_0^2\varphi^2+\tfrac{1}{2}\beta N\hbar\delta(0)\ln N^{-1}\varphi^2. \tag{2.16}$$

This is the disastrous potential of (1.8) that we wish to avoid.

We shall show that the quantum effects ($\hbar\neq 0$) in (2.14) and (2.15) are sufficient to avert the danger.

The global extremum of $\mathcal{V}(\varphi)$ occurs at

$$0=\partial\mathcal{V}/\partial\varphi_i=m^2\varphi_i \tag{2.17}$$

i.e. $\boldsymbol{\varphi}=\mathbf{0}$. At $\boldsymbol{\varphi}=\mathbf{0}$ it follows, from (2.14) and (2.15), that

$$\sigma(0)=\hbar\int\frac{đk}{k^2+m^2(0)}=\hbar G(0,m^2(0)) \tag{2.18}$$

$$m^2(0)=m_0^2+\frac{\beta\delta(0)}{G(0,m^2(0))}. \tag{2.19}$$

Equation (2.19) was discussed extensively in III for $\beta=1$, where it was shown that it could only be expressed in terms of finite quantities for $d\geqslant 4$ space–time dimensions. This is equally true for all $\beta>0$.

As in III we introduce a momentum cut-off $|k|<\Lambda$. For simplicity we restrict ourselves to $d>4$ dimensions†.

Let us develop $\mathcal{V}(m^2,\sigma;\varphi)$ as an expansion in φ^2,

$$\tilde{m}^2=m^2(\varphi^2)-m^2(0)\qquad\qquad \tilde{\sigma}=\sigma(\varphi^2)-\sigma(0). \tag{2.20}$$

This gives (up to a constant)

$$\mathcal{V}(m^2,\sigma;\varphi)=\tfrac{1}{2}m^2(0)\varphi^2+\tfrac{1}{2}\tilde{m}^2\varphi^2-\tfrac{1}{4}\beta N\hbar\delta(0)\sigma(0)^{-2}\tilde{\sigma}^2-\tfrac{1}{2}\tilde{m}^2\tilde{\sigma}$$
$$-\frac{N\hbar}{4}(\tilde{m}^2)^2\int\frac{đk}{(k^2+m^2(0))^2}+\text{higher-order terms.} \tag{2.21}$$

Expressed in powers of Λ, the coefficients of the higher-order terms get progressively less singular.

On imposing the constraints (2.19) and (2.20) we find that in terms of

$$B(m^2)=\int\frac{đk}{(k^2+m^2)^2} \tag{2.22}$$

we have

$$\tilde{\sigma}=\frac{\varphi^2}{N}\left(1-\frac{\beta\hbar^2\delta(0)}{\sigma(0)^2}B(m^2(0))\right)^{-1}+\mathrm{O}((\varphi^2/N)^2) \tag{2.23}$$

$$\tilde{m}^2=-\frac{\varphi^2}{N}\frac{\hbar\delta(0)}{\sigma(0)^2}\left(1-\frac{\beta\hbar^2\delta(0)}{\sigma(0)^2}B(m^2(0))\right)^{-1}+\mathrm{O}((\varphi^2/N)^2). \tag{2.24}$$

† The case $d=4$ will be considered elsewhere. It is sufficient for our purposes here to demonstrate that stability can be preserved in principle.

Again, expressed in powers of Λ the first terms in (2.23) and (2.24) are the most singular. With

$$\delta(0)=\mathrm{O}(\Lambda^{d}) \qquad \sigma(0)=\mathrm{O}(\Lambda^{d-2}) \qquad B(m^2(0))=\mathrm{O}(\Lambda^{d-4}) \tag{2.25}$$

we see that

$$\tilde{\sigma}=\frac{\varphi^2}{N}\mathrm{O}(1) \qquad \tilde{m}^2=\frac{\varphi^2}{N}\mathrm{O}(\Lambda^{4-d}). \tag{2.26}$$

Taking the terms in $\mathscr{V}$ in (2.22) in order, this gives

$$\begin{aligned} &\tfrac{1}{2}\tilde{m}^2\varphi^2=\frac{(\varphi^2)^2}{N}\mathrm{O}(\Lambda^{4-d}) \qquad &&\delta(0)\frac{\tilde{\sigma}^2}{\sigma(0)^2}=\frac{(\varphi^2)^2}{N}\mathrm{O}(\Lambda^{4-d}) \\ &\tilde{m}^2\tilde{\sigma}=\frac{(\varphi^2)^2}{N}\mathrm{O}(\Lambda^{4-d}) \qquad &&(\tilde{m}^2)^2B(m^2(0))=\frac{(\varphi^2)^2}{N}\mathrm{O}(\Lambda^{4-d}). \end{aligned} \tag{2.27}$$

The higher-order terms omitted are down by additional powers of Λ. Thus, in the limit $\Lambda\to\infty$ for $d>4$ dimensions only the first term in (2.22) survives to give

$$\mathscr{V}(\varphi)=\tfrac{1}{2}m^2(0)\varphi^2 \tag{2.28}$$

where $m^2(0)$ satisfies (2.20).

That is, in the large-N limit the pseudo-free theory is in fact a (stable) *free* theory in $d>4$ dimensions.

The mechanism that has brought about this stability is that the large-N limit resums the diagrams associated with the 'hard-core' change of measure in such a way that their contribution vanishes on removing the Λ cut-off, provided the individual terms are sufficiently singular.

The fact that the large-N limit of the pseudo-free theory is a stable free theory does not mean that, in general, it is so. The non-leading terms in a $1/N$ expansion will provide non-free corrections. However, our empirical experience of $1/N$ expansions suggests that such pathologies as they possess are present at leading order. We do not expect non-leading orders to reintroduce instability, particularly as the singularity of the measure was a leading-order effect. At the moment the systematic development of the $1/N$ expansion is under active study and we shall report on it elsewhere. Similarly, as we have already mentioned, we do not expect the inclusion of self-interactions to alter the stability†.

We think that this approach is the most promising way to solve for scale-covariant theories. However, our understanding of scale-covariant theories is still sufficiently poor that all avenues should be explored. All that is clear is that $\hbar$ expansions are inappropriate and, in consequence, any non-perturbative (in $\hbar$) approach of canonical theory may have some use in scale-covariant theory.

3. Expansion in kinetic terms

Let us revert to the theory of a *single* pseudo-free scalar field. Because the effective potential $\mathscr{V}(\varphi)$ only depends on constant field strengths, kinetic terms are absent in

† This particular aspect of stability will be pursued in paper V in this series (Ebbutt and Rivers 1982c), where we examine the large-N limit of the O(N) self-interacting scale-covariant $\lambda(\varphi^2)^2$ theory.

the first instance†. This suggests that an alternative approach may be to develop a power series in the kinetic terms about the theory with kinetic-free action

$$A_0[\varphi] = \int \mathrm{d}x \, (\tfrac{1}{2} m_0^2 \varphi^2). \tag{3.1}$$

The independent-value model (IVM) with action A_0 is exactly, and non-trivially, solvable (Klauder 1975, 1979a, b). In particular, it requires a scale-*invariant* measure $\beta = 1$.

At one extreme it has been argued (Kovesi-Domokos 1976) that such a 'strong-coupling' expansion (or, more appropriately, a large m^2 expansion) gives rise to a semiclassical theory of tree diagrams, with the vertices and mass terms of the IVM. If this were true it would mean that the Euclidean effective potential $\mathscr{V}(\varphi)$ for the theory would be just that obtained from the IVM, and it is this that we shall now calculate.

As we saw in II, the IVM does not permit any mass renormalisation like (2.20). Rather, the scale covariance forces a multiplicative renormalisation of the form

$$m^2 = b^{-1} \delta(0) m_0^2 \tag{3.2}$$

where b, with dimension $[\text{Mass}]^d$, is an arbitrary mass scale.

Let $Z_0'[j]$ be the generating functional for the IVM,

$$Z_0'[j] = \int \mathscr{D}'[\varphi] \exp -\frac{1}{\hbar} \int \mathrm{d}x \, (\tfrac{1}{2} m_0^2 \varphi^2 - j\varphi) \tag{3.3}$$

whose effective potential we wish to calculate. As before, we need only consider constant j, whence

$$Z_0'[j] = \exp -\frac{\Omega}{\hbar} W_0'(j) \tag{3.4}$$

where (Klauder 1975)

$$W_0'(j) = -\alpha b \hbar \int_0^\infty \frac{\mathrm{d}u}{u} [\cosh(uj/\hbar) - 1] \exp -(bm^2 u^2 / 2\hbar). \tag{3.5}$$

The coefficient αb is an undetermined scale factor (see II) and, for convenience, we set $\alpha = 1$.

The Euclidean effective potential $\mathscr{V}(\varphi)$ is the Legendre transform of $W(j)$. We define

$$\begin{aligned} \varphi &= -\frac{\partial W_0'}{\partial j} = b \int_0^\infty \mathrm{d}u \, \sinh(uj/\hbar) \exp -(bm^2 u^2 / 2\hbar) \\ &= \left(\frac{\pi b \hbar}{2m^2}\right)^{1/2} \exp\left(\frac{j^2}{2b\hbar m^2}\right) \Phi\left(\frac{j}{m\sqrt{2b\hbar}}\right) \end{aligned} \tag{3.6}$$

where Φ is the error function. We then have

$$\mathscr{V}(\varphi) = W_0'(j(\varphi)) + \varphi j(\varphi) \tag{3.7}$$

whence

$$\mathscr{V}(\varphi) = \int_0^\varphi j(\varphi') \, \mathrm{d}\varphi'. \tag{3.8}$$

† Of course, in any expansion scheme the kinetic terms will determine the form of the loop contributions.

From the monotonicity of $\varphi(j)$ it follows that $\mathcal{V}(\varphi)$ is bounded below by zero. That is, the Euclidean effective potential shows no signal for instability.

We can say more. The scale is set by the quantity $b\hbar/m^2$. For $\varphi^2 \ll b\hbar/m^2$ we have

$$\mathcal{V}(\varphi) = \tfrac{1}{2}m^2\varphi^2 + m^4\varphi^4/12b\hbar + O(\varphi^6/(b\hbar)^2).$$

But for $\varphi^2 \gg b\hbar/m^2$,

$$\mathcal{V}(\varphi) \sim \varphi\sqrt{2b\hbar m^2}[\ln \varphi(2m^2/\pi b\hbar)^{1/2}]^{1/2}. \tag{3.9}$$

Thus, as $b\hbar/m^2 \to \infty$ we recover the *free* theory. This can be interpreted in two ways. On the one hand, it says that at mass scales that are small compared with b we cannot distinguish the pseudo-free from the free theory. This was noticed in (Klauder 1979a, b), but it has been stated (Nouri-Moghadam and Yoshimura 1978) that this is a particular example of a more general property of scale covariance. That is, that scale-covariant theories possess mass scales below which they behave like canonical theories. This may be relevant to theories of grand unification, which naturally give rise to large mass scales. The above comments suggest that in some sense, it does not matter whether the theory is renormalisable or not, provided we are at energies well below these scales.

Alternatively we could interpret $b\hbar$ large as a high-temperature limit for the theory†.

Of course, all the above presupposes that the effective potential $\mathcal{V}(\varphi)$ for the IVM is indeed the effective potential of the whole theory, which would only be true if the analytic regularisation procedure of Kovesi-Domokos (1976) was exact. We do not believe this to be the case‡. Rather, we expect something more complicated to be at work (e.g. see Caianiello *et al* (1978), Bender *et al* (1980) for discussions of strong-coupling expansions). Nonetheless, treating Kovesi-Domokos (1976) as an approximation it certainly describes part of the picture, and perhaps a significant part at high temperatures, for example.

We have primarily introduced this approximate 'strong-coupling' calculation to demonstrate that more than one mechanism exists to preserve stability in scale-covariant theories. In this case it is that the operator-product expansion implicit in the scale-covariant formalism essentially expresses $W'(j)$ rather than $Z'(j)$, in terms of exponential integrals with singular measures. In consequence the semiclassical field $\varphi[j]$ has a very different behaviour with respect to j from that of a canonical theory. Unfortunately, it is not possible to relate this to the large-N limit of the previous section because the large-N limit of the $O(N)$-invariant IVM is somewhat pathological (Klauder and Narnhoffer 1976)§.

4. The Minkowski theory

So far we have restricted ourselves to the Euclidean pseudo-free theory. We have construed the lower boundedness of the Euclidean effective potential $\mathcal{V}(\varphi)$ as a signal for the stability of the theory. This is not strictly true, since it is only for the Minkowski theory that the effective potential has the interpretation of an energy density (see, for example, Coleman 1975).

† Both these interpretations are very unlike the large-N limit of the previous section.

‡ The fact that our conclusions are dimension independent makes us suspicious.

§ This will become clearer in paper V.

For the large-N limit of the O(N)-invariant pseudo-free theory there is no difficulty in performing the analytic continuation from the Minkowski to the Euclidean theory. The Minkowski effective potential is still ($d>4$)

$$\mathscr{V}(\varphi)=\tfrac{1}{2}m^2(0)\varphi^2 \tag{4.1}$$

where $m^2(0)$ is given by (2.19).

For the expansion about the IVM the situation is different. If the Euclidean tree diagrams are all that survive, continuation is trivial and nothing changes. However, we have already observed that the over regularisation of Kovesi-Domokos (1976) is too extreme. This is reflected in the fact that other continuations are possible with this regularisation that lead to different conclusions. Without giving them any particular credence, it is worthwhile summarising them briefly.

For example, we might wish to begin with the overtly Minkowski IVM generating functional

$$Z_0''[j]=\int \mathscr{D}'[\varphi]\exp -\frac{\mathrm{i}}{\hbar}\int \mathrm{d}x(\tfrac{1}{2}m_0^2\varphi^2-j\varphi). \tag{4.2}$$

This gives (for constant j)

$$Z_0''(j)=\exp \mathrm{i}\Omega W_0''(j)/\hbar \tag{4.3}$$

with

$$W_0''(j)=\mathrm{i}b\hbar\int_0^\infty \frac{\mathrm{d}u}{u}(1-\cos uj/\hbar)\exp-(\mathrm{i}bm^2u^2/2\hbar). \tag{4.4}$$

The most immediate consequence of (4.4) is that

$$\varphi=\partial W_0''/\partial j \tag{4.5}$$

is no longer single valued (in magnitude at least), with

$$\varphi\to 0 \qquad \text{as} \qquad j\to 0 \qquad \textit{and} \qquad j\to\infty. \tag{4.6}$$

However, on inspection we see that real φ requires complex j (and vice versa) whence $\mathscr{V}(\varphi)$ is also complex. It might be argued that this complexity should be understood as a measure of the instability of the vacuum (with decay rate proportional to Im $\mathscr{V}$). Rather, we interpret it as an overt failure of the regularisation of Kovesi-Domokos (1976).

Nonetheless, this suggests an alternative continuation in which we commence with the Euclidean IVM for which

$$Z_0'''[j]=\int \mathscr{D}'[\varphi]\exp -\frac{1}{\hbar}\int \mathrm{d}x[\tfrac{1}{2}m_0^2\varphi^2-\mathrm{i}j\varphi] \tag{4.7}$$

and continue afterwards. This differs from (3.2) in the replacement of j by ij and is, in fact, the form used in paper II. On taking j constant we have

$$Z_0'''(j)=\exp -\Omega W_0'''(j)/\hbar \tag{4.8}$$

where

$$W_0'''(j)=b\hbar\int_0^\infty \frac{\mathrm{d}u}{u}(1-\cos uj/\hbar)\exp-(bm^2u^2/2\hbar). \tag{4.9}$$

The semiclassical constant field $\varphi = \partial W_0'''/\partial j$ is expressible in terms of confluent hypergeometric functions as

$$\varphi = \frac{\hbar b j}{m^2} M(1, \tfrac{3}{2}; -j^2/2b\hbar m^2). \tag{4.10}$$

As in the original Euclidean example, φ is real but, as in the Minkowski example above, is double valued. In consequence the effective potential is double branched. In the vicinity of $\varphi = 0$ the upper branch $\mathscr{V}_u(\varphi)$ is as the Euclidean potential $\mathscr{V}(\varphi)$ of (3.8).

However, on the lower branch ($j \to \infty$)

$$j \sim b\hbar\varphi^{-1} \tag{4.11}$$

whence

$$\mathscr{V}_l(\varphi) \sim \tfrac{1}{2} b\hbar \ln \varphi^2 \tag{4.12}$$

for small φ. That is, the disturbing logarithmic singularity (in regularised form, with $\delta(0)$ replaced by b) has reappeared on the lower branch. As $b\hbar \to \infty$ the two branches decouple, but for $b\hbar$ finite they could communicate via instantons if the regularisation (Kovesi-Domokos 1976) were exact.

Effective potentials with more than one branch are not unknown†. However, once instantons cease to be driven by the classical kinetic terms (of purely tree diagrams) tunnelling is no longer inevitable (Cant 1979). It is likely that a more correct regularisation would show this. Alternatively, a phase transition could exist below which the lower branch decoupled or vanished.

Given the supporting evidence of the large-N limit we are satisfied to see that the operator-product expansion of the scale-covariant formalism is able, in principle, to provide a mechanism for stability. We are unable to pursue this 'strong-coupling' approach any further at the moment.

5. Conclusion

We have presented two mechanisms for preserving the stability of scale-covariant scalar theories, despite the presence of $\ln \varphi^2$ terms in the 'classical' potential induced by the change of measure.

Firstly, we can resum the diagrams due to the 'hard-core' effect of the change of measure as in the large-N limit of the $O(N)$-invariant theory (or equivalently, perform a mean-field expansion). For the pseudo-free theory we have seen that, on removing the momentum-space cut-off, the hard-core effects vanish in the large-N limit for $d > 4$ dimensions. This dimension-specific result is encouraging, since for $d > 4$ dimensions the scale-covariant formalism is forced upon us. Stability is thus restored to leading order. While not proving stability, this makes it much more probable.

Alternatively, we have examined a 'strong-coupling' series expansion in kinetic terms about the independent-value model in an approximation in which only tree diagrams survive. The operator-product expansion implied by the scale covariance essentially forces W (the generating functional for connected Green functions) to play

† For example, the double-branched effective potential of the large-N theory has a real part which is unbounded below (see Cant 1979).

the role that Z (the generating functional for unconnected Green functions) plays in canonical theories. In this way the effective potential acquires a lower bound (for the Euclidean theory).

In this latter case there is some ambiguity in continuation from the Euclidean to the Minkowski theory but stability can be preserved.

We conclude that, as yet, there is no problem with the stability of scale-covariant theories. The next step is to examine non-leading terms in the $1/N$ expansion of the $O(N)$ scalar theory. This will be reported elsewhere.

Note added in Proof. In our analysis of the stability of the large-N $O(N)$-invariant pseudo-free theory in § 2, we assumed that, on imposing ultraviolet regularisation $|\boldsymbol{k}| \leq \Lambda$, the expression

$$\left\{1-\frac{\beta\hbar^2\delta(0)}{\sigma(0)^2}B(m^2(0))\right\}$$

is $O(\Lambda^0)$. This is true for general values of β, but if β is chosen carefully it can vanish as $\Lambda\to\infty$. Equations (2.23)–(2.27) then cease to be valid. Nonetheless, although different, if we calculate $\mathscr{V}(\varphi^2)$ in the vicinity of $\varphi^2=0$ we still have *no* instability. However, although our conclusions about instability are unaltered, the way in which stability is restored is changed. For these particular values of β the large-N limit is no longer that of a free theory. The consequences of this will be discussed elsewhere by one of us (RJR).

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